

Localization of the states of a PT -symmetric double well

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Abstract

We make a nodal analysis of the processes of level crossings in a model of quantum mechanics with a PT -symmetric double well. We prove the existence of infinite crossings with their selection rules. At the crossing, before the PT -symmetry breaking and the localization, we have a total P -symmetry breaking of the states.

1 Introduction

The interest on simple quantum mechanical models is also given by certain similarities with quantum field theory. In particular, is of great interest the possibility of summing divergent perturbation series. This problem is related to the existence of singularities of the levels as functions of the perturbation parameter. Such singularities, due to the level crossings, are not so easy to study in a rigorous way. The semiclassical theory provided good qualitative and quantitative results for lower semiclassical parameter up to the crossing

value [3]-[6], [15], [10]. The exact semiclassical method [13] has given good qualitative and quantitative results for larger values of the parameter [11], [12]. We believe that only the nodal analysis, begun in the papers [7], [8], [9], [2], can give a clear and exhaustive analysis of the level crossings. In recent times the main interest focused on the PT -symmetric Hamiltonians [6], [16]. In particular it was of some interest to prove the reality and the analyticity of the spectrum of certain oscillators [1], [18]. André Martinez and one of us (V. G.) in the paper [2] have proved the Padé summability of the perturbation series to the perturbative levels $\tilde{E}_n(\beta)$ of the imaginary cubic oscillator,

$$H(\beta) = p^2 + x^2 + i\sqrt{\beta}x^3, \quad p^2 := -\frac{d^2}{dx^2}, \quad \beta \neq 0, \quad |\arg(\beta)| < \pi. \quad (1)$$

In this paper [2], was also used the semiclassical method, but the exact results was mostly given by the control of the nodes of the states. Our program is to extend the analysis of the perturbative levels to the other regions of β where the level crossings are expected.

By changing representation, we study the spectrum of a semiclassical Hamiltonian, the closed, PT -symmetric operator,

$$H_{\hbar} := \hbar^2 p^2 + V(x), \quad V(x) := i(x^3 - x), \quad \hbar > 0, \quad (2)$$

where the derivative of the potential $V'(x)$ has two real zeros: $x_{\pm} = \pm 1/\sqrt{3}$. In some sense, (2) is a PT -symmetric double well Hamiltonian. The large \hbar behavior of the levels $E_n(\hbar)$, corresponding to the perturbative levels $\tilde{E}_n(\beta)$, is studied by the other Hamiltonian,

$$K(\alpha) = p^2 + W(\alpha, x), \quad W(\alpha, x) = i(x^3 + \alpha x), \quad \alpha \geq 0. \quad (3)$$

The level $\hat{E}_m(\alpha)$, $m \in \mathbb{N}$, of $K(\alpha)$ is holomorphic on the sector,

$$\mathbb{C}_{\alpha} := \{\alpha \in \mathbb{C}; \alpha \neq 0, |\arg(\alpha)| < 4\pi/5\}, \quad (4)$$

but can be analytically continued to $-\alpha_n > -\alpha > 0$ small, up to the first crossing at α_n , where we define directly the level of H_{\hbar} , by the relation,

$$E_m(\hbar) = \hbar^{6/5} \hat{E}_m(\alpha), \quad m \in \mathbb{N}, \quad \alpha = -\hbar^{-4/5}. \quad (5)$$

The level $\hat{E}_m(\alpha)$, real analytic for $\alpha > 0$ [1], [2], is real analytic also for $-\alpha > 0$ small. Since we know the absence of singularities of $E_n(\hbar)$ for small

$|\hbar|$, or $\hat{E}_n(\alpha)$ for large $|\alpha|$ [18], in certain sectors, we define two other types of levels for small $\hbar > 0$, by the analytic continuation of $\hat{E}_n(\alpha)$ on the complex plane, along paths starting from $\alpha = \hbar^{-4/5}$, continuing with $|\alpha| = \hbar^{-4/5}$, and arriving to $\alpha^\pm := \exp(\pm i\pi)\hbar^{-4/5}$, respectively. Thus, we define the levels,

$$E_n^\pm(\hbar) := \hbar^{6/5} \hat{E}_n(\alpha^\mp), \quad n \in \mathbb{N}, \quad \alpha^\pm = \exp(\pm i\pi)\hbar^{-4/5}, \quad \hbar > 0. \quad (6)$$

All such levels are extendible as multi-valued functions, to the sector on the \hbar complex plane:

$$\mathcal{C}^0 = \{\hbar \in \mathbb{C}; \hbar \neq 0, |\arg(\hbar)| < \pi/4\}. \quad (7)$$

We extend all the states,

$$\psi_n^\pm(\hbar, z), \quad n \in \mathbb{N}, \quad \hbar > 0 \text{ small}, \quad \psi_m(\hbar, z), \quad m \in \mathbb{N}, \quad \hbar > 0 \text{ large}, \quad z \in \mathbb{C},$$

for fixed \hbar , as entire functions on the complex z plane. For large fixed $\hbar > 0$, the representation of the state $\psi_m(z)$ is taken $P_x T$ -symmetric, where $P_x \psi_m(x + iy) = \psi_m(-x + iy)$. In this case, the set of the nodes, as the set of the other zeros, of $\psi_m(z)$ is P_x symmetric. For small fixed $\hbar > 0$, we have $\psi_n^+(z) = P_x T \psi_n^-(z)$, so that the set of nodes S_n^\pm of the state ψ_n^\pm is the P_x transform of S_n^\mp . For small $\hbar > 0$, both the levels $E_n^\pm(\hbar)$, $n \in \mathbb{N}$, $E_n^+(\hbar) = \bar{E}_n^-(\hbar)$, are non-real up to the first crossing. In particular, the perturbative levels have the following semiclassical behavior,

$$E_n^\pm(\hbar) = \mp i \frac{2}{3\sqrt{3}} + \sqrt{\pm i} \sqrt[4]{3} (2n+1) \hbar + O(\hbar^2), \quad \hbar > 0, \quad (8)$$

in the limit $\hbar \rightarrow 0^+$. These behaviors correspond to the behaviors of the perturbative levels of the Hamiltonian (1). In the same limit, all the nodes of the states $\psi_n^\pm(\hbar)$ shrink to the centers of the wells,

$$x_\pm \in \mathcal{C}^\pm = \{z \in \mathbb{C}; \pm \Re z > 0\},$$

respectively. Now, we prove that the zeros, and in particular the nodes, of a state $\psi_n^\pm(\hbar)$ cannot reach and cross the imaginary axis (Lemma 5). Moreover, the nodes on \mathcal{C}^\pm respectively, cannot disappear going to infinite (Lemma 4). This means that the number n of the nodes of a state $\psi_n^\pm(\hbar)$ in \mathcal{C}^\pm are stable for small \hbar up to the first crossing at $h_n = (-\alpha_n)^{-5/4} > 0$. At the limit of $\hbar \rightarrow h_n^-$, the energy levels $E_n^\pm(\hbar)$ have the limit $E_n^c > 0$ and the set of

the zeros of $\psi_n^\pm(\hbar)$ becomes P_x -symmetric. The critical state ψ_n^c , has a P_x -symmetric set of $2n$ nodes in $C^{+-} := C^+ \cup C^-$ (Lemma 7). The discontinuity of the number of nodes at h_n , $n > 0$, is due to the PT -symmetry breaking of the states.

Now we look for a pair of positive analytic levels $E_j(\hbar), E_k(\hbar)$, $j, k \in \mathbb{N}$, $j \neq k$, with limits $E_j(h_n^+) = E_k(h_n^+) = E_n^c$ and corresponding states with the limit value $\psi_j(h_n^+) = \psi_k(h_n^+) = \psi_n^c$. This is possible if both the corresponding P_x -symmetric sets of nodes $S_j(\hbar), S_k(\hbar)$ contain $2n$ nodes stable in C^{+-} . But what can be said about the imaginary nodes? In order to distinguish a possible node in the imaginary axis from the other zeros, we consider the limit $\hbar \rightarrow +\infty$ corresponding to the limit $\alpha \rightarrow 0^-$. We prove (Lemma 6) that the nodes in this limit are confined in the lower half plane. Since the imaginary turning point, for a level $E > 0$, is $I_0 = i\tilde{y}$, $\tilde{y} > 0$, we define as an imaginary node a zero in $\Sigma = i(-\infty, \tilde{y})$. We also prove that the number of imaginary nodes can be zero or one. Since there are in any case $2n$ non imaginary nodes, the two independent states are necessarily the states $(\psi_{2n}(\hbar), \psi_{2n+1}(\hbar))$, with levels $(E_{2n}(\hbar) < E_{2n+1}(\hbar))$ for $\hbar > h_n$. Actually, exists $h_n^p > h_n$ such that the state $\psi_{2n+1}(\hbar)$ has an imaginary node for $\hbar > h_n^p$. In Lemma 9 we give the crossing selection rules (24):

$$E_n^\pm(h_n^-) = E_m(h_n^+) := E_n^c > 0,$$

$$\psi_n^\pm(h_n^-) = \psi_m(h_n^+) := \psi_n^c, \quad \forall m \in \mathbb{N}, \quad [m/2] = n. \quad (9)$$

The critical state ψ_n^c is orthogonal to its P -transform, or, in other words, is totally P -asymmetric (Lemma 10). The crossing corresponds to a square root singularity of this pair of analytic functions, positive analytic for $\hbar > h_n$. For the value of h_n^p and $E_{2n+1}(h_n^p) := E_n^p$, we have only numerical results. Our numerical computations (Table 1) show that $h_n^p \rightarrow 0$, and $E_{2n+1}(h_n^p) := E_n^p \rightarrow E^p > 0$ as $n \rightarrow \infty$ with $E_n^p - E^p = O((h_n^p)^2)$, where $E^p \sim 0,352268..$ is the unique energy such that the imaginary turning point [12] is on the short Stokes line [8], [9]. The semiclassical state corresponding to the energy E^p is considered a bilocalized state, the transition state between the delocalized and the localized state.

The unicity of the crossing for each pair E_n^\pm is taken as a conjecture (Conjecture 1) assumed in order to simplify the notations and the discussion.

We give the structure of the Riemann sheet of the levels $E_m(\hbar)$ from large $\hbar > 0$ to all the real axis, with the values at the borders of the cut $(0, h_n]$

(Theorem 1).

In Sec. 2 we prove the positivity of the spectrum for large \hbar and the reality of the states on the imaginary axis; in Sec. 3 we consider the appearance of an imaginary node for the odd states; In Sec. 4, we follow the process of crossing; in Sec. 5 we prove that for small $\hbar > 0$ the imaginary axis is free of zeros and the nodes are bounded; in Sec. 6 we prove a confinement of the nodes for large $\hbar > 0$; in Sec. 7 we prove the quantization rules, the continuity and the boundedness of the levels; in Sec. 8 we prove the total P-symmetry breaking at the crossing; in Sec. 9 we give the local structure of the Riemann sheets of the positive levels with the cuts directed toward 0.

n	h_n^p	E_n^p
8	0.043835	0.3519 ± 0.0010
9	0.030683	0.3514 ± 0.0011
10	0.023605	0.3518 ± 0.0013
11	0.013060	0.3522 ± 0.0002

Table 1: The values of h_n^p , and E_n^p with the errors, for different values of $n = 8 - 11$.

2 Positivity of the levels and reality of the states on the imaginary axis for large $\hbar > 0$

The level $\hat{E}_m(\alpha)$, $m \in \mathbb{N}$ of $K(\alpha)$ is analytic in a neighborhood of the origin $U \subset \mathbb{C}$ [2], [20]. Since it is real analytic for $\alpha < 0$ it is real analytic also in $U \cup \mathbb{R}$ [1]. The positivity comes from the positivity of the kinetic energy,

$$\Re \hat{E}_m(\alpha) = \Re \langle \hat{\psi}_m(\alpha), K(\alpha) \hat{\psi}_m(\alpha) \rangle = \langle \hat{\psi}_m(\alpha), p^2 \hat{\psi}_m(\alpha) \rangle > 0,$$

where $\psi_m(\alpha)$ is the corresponding normalized state. Also the level $E_m(\hbar)$ is real analytic and positive for $\hbar > 0$ large enough. Thus, we have proved:

Lemma 1

The level $E_m(\hbar)$, $m \in \mathbb{N}$, is real analytic and positive for $\hbar > 0$ large enough.

We now extend the analysis of the analytic states on the complex plane.

Let us consider $y \in \mathbb{R}$ and the translation $f(x) \rightarrow f(x + iy)$, so that the PT -symmetric Hamiltonian becomes the other isospectral PT -symmetric Hamiltonian on the \mathcal{H}_y representation:

$$H_{\hbar}(y) := \hbar^2 p^2 + i(x^3 - (3y^2 + 1)x) - (3yx^2 - y^3 - y) \sim H_{\hbar}. \quad (10)$$

The eigenfunction $\psi_{n,y}(x) := \psi_n(x + iy)$ on the \mathcal{H}_y representation, with real eigenvalue E_n , can be taken PT -symmetric:

$$PT\psi_{n,y}(x) := \overline{\psi}_{n,y}(-x), \quad (11)$$

and in particular

$$\psi_{n,y}(0) = \overline{\psi}_{n,y}(0) = \psi(iy).$$

Thus, we have proved the following,

Lemma 2

For large $\hbar > 0$, the level $E_m(\hbar)$, $m \in \mathbb{N}$, is positive and, for a choice of the gauge, the state $\psi_m(\hbar)$, extended to the complex plane as an entire function, is $P_x T$ -symmetric:

$$(P_x T\psi_m)(x + iy) := \overline{\psi}_m(-x + iy) = \psi_m(x + iy), \quad \forall x, y \in \mathbb{R}, \quad (12)$$

and, in particular, the state is real on the imaginary axis,

$$\Im\psi_m(iy) = 0, \quad \forall y \in \mathbb{R}. \quad (13)$$

The set of all the zeros of the state is P_x -symmetric.

3 The nodal analysis of the process of crossing

Let $E_m(\hbar)$, for $\hbar > 0$ large enough, be a positive level of the Hamiltonian (2) with a corresponding state $\psi_m(\hbar)$. Now, by the complex dilation $z \rightarrow iz$, we consider the Hamiltonian on the imaginary axis:

$$H_{\hbar}^r = -\hbar^2 \frac{d^2}{dy^2} + \tilde{V}(y) \sim -H_{\hbar}, \quad \tilde{V}(y) := -y^3 - y, \quad (14)$$

well defined by the L^2 condition on the x -axis, here playing the role of the imaginary axis. The Hamiltonian H_{\hbar}^r has the same spectrum as $-H_{\hbar}$, so

that $-E := -E_m(\hbar) < 0$ is one of its eigenvalues. The corresponding state $\phi_m(y) := \psi_m(iy)$ can be taken real. Actually, since we can have the $P_x T$ -symmetry of the entire state $\psi_m(z)$, we can have the reality of $\psi(iy)$ for real y : $\psi_m(x+iy) = \bar{\psi}_m(-x+iy)$, $\psi_m(iy) = \bar{\psi}_m(iy)$. We consider together the two states $\psi_m(z)$, $[m/2] = n \in \mathbb{N}$, for a fixed $\hbar \geq h_n$ (24). Both the states have n nodes on both the half-planes \mathbb{C}^\pm and are distinguished by the number of imaginary nodes for $\hbar > 0$ large. All the process of crossing for $\hbar \geq h_n$ can be studied by the behaviors of the states $\psi_m(z)$, with energy $E = E_m$, $[m/2] = n$ on the imaginary semi-axis,

$$\Sigma(E) = \{z = iy; -\infty < y < \tilde{y}(E)\}, \quad (15)$$

where the imaginary turning point is $I_0 = i\tilde{y}(E)$. This means that we consider the two states $\phi_m(y)$, of (14), with energies $-E = -E_m$, $[m/2] = n$ for $y \leq \tilde{y}(E)$.

For $\hbar > 0$ large, we have two possible behaviors of the state $\phi(y)$ of (14) with level $-E$. Let us recall that if, for y in a bounded interval of the semi-axis $-\infty < y < \tilde{y}(E)$, a state $\phi(y)$ is positive, it is convex; if it is negative, it is concave. On the other side, for $y > \tilde{y}(E)$, where an eigenfunction $\phi(y)$ is positive it is also concave, and where it is negative it is also convex.

Since we can consider $\phi(y)$ positive decreasing for $y \ll \tilde{y}(E)$, there are only two cases:

- a) the existence of one node on $\Sigma(E)$,
- b) the absence of nodes on $\Sigma(E)$.

Let us remark that $\tilde{y}(E) > 0$ so that a possible node on the imaginary axis should be in $\Sigma(E)$ for large \hbar . Thus, we have the result:

Lemma 3

The state $\psi_m(\hbar)$, $m \in \mathbb{N}$, with corresponding positive level $E = E_m(\hbar)$, have at most one zero in $\Sigma(E)$. This zero, considered a node, exists for $\hbar > h_n$ large enough if $m = 2n + 1$, but don't exists if $m = 2n$.

4 Non real levels: imaginary axis free of zeros and bounded nodes

Let us consider the general case with a level $E \in \mathbb{C}$, and the corresponding state $\psi(z)$, with $\psi(iy) = \phi(y)$, $z, y \in \mathbb{C}$. We transform the Hamiltonian by

imaginary translations:

$$H_h^r(x) = -\hbar^2 \frac{d^2}{dy^2} + \tilde{V}(y - ix),$$

$$\begin{aligned} \tilde{V}(y - ix) &= -(y - ix)^3 - (y - ix) = -y^3 + 3x^2y - y + i(x(3y^2 + 1) - x^3) = \\ &= \Re \tilde{V}(y - ix) + i\Im \tilde{V}(y - ix), \end{aligned}$$

where $\Im \tilde{V}(y - ix) = (x(3y^2 + 1) - x^3)$ with level $-E$, for a fixed $x \neq 0$, and we consider a state,

$$\phi_x(y) := \phi(y - ix), \quad n \in \mathbb{N},$$

with the well known asymptotic behaviors,

$$|\phi_x(y)|^2 \sim C|y|^{-3/2} \text{ for } y \rightarrow +\infty,$$

$$|\phi_x(y)|^2 \sim C|y|^{-3/2} \exp(-2y^{5/2}/\hbar) \text{ for } y \rightarrow -\infty. \quad (16)$$

We apply the Loeffel-Martin method [19] generalized to the case of diverging integrals:

$$\begin{aligned} &\hbar^2 \Im [\bar{\phi}_x(y) \partial_y \phi_x(y)] = \\ &= \hbar^2 \Im [\bar{\phi}_x(\tilde{y}) \partial_y \phi_x(\tilde{y})] + \int_{\tilde{y}}^y (x(3s^2 + 1) - x^3 + \Im E) |\phi_x(s)|^2 ds \rightarrow +\infty, \end{aligned} \quad (17)$$

where $x(3s^2 + 1) - x^3 = \Im \tilde{V}(s - ix)$, as $y \rightarrow +\infty$ for fixed $\tilde{y}, x \in \mathbb{R}, x \neq 0$. We know that the zeros, for $|z|$ large, have the asymptotic direction $\arg z \rightarrow \pi/2$ [2]. By (17) we prove a stronger condition on the asymptotics of the zeros:

Lemma 4

Let E be a level with state $\psi(z)$ of the Hamiltonian H_h for a fixed $\hbar > 0$. Consider a generic zero $Z_j = X_j + iY_j$ of $\psi(z)$. Exists $M > 0$, such that $\pm X_j > 0$ if $\mp \Im E > 0$, $Y_j > M$. In case of real level, $\Im E = 0$, the large zeros are purely imaginary.

proof

The integral in (17) don't diverge for $y \rightarrow +\infty$ only if x depends on y such that,

$$x(y) \rightarrow 0, \quad \pm x(y) \geq \frac{|\Im E|}{3y^2 + 1}, \quad \mp \Im E > 0, \quad \text{as } y \rightarrow +\infty. \quad (18)$$

Actually, condition (18) is necessary for having a change of sign on the integrand in (17). Otherwise, the integral in (17) diverge. We have the state $\phi_E(y) := \psi_E(iy)$ with corresponding level E of the Hamiltonian H_{\hbar} . We consider the Loeffel-Martin formula in order to generalize to our problem the expression of the imaginary part of a shape resonance:

$$\hbar^2 \Im(\bar{\phi}(y) \partial_y \phi(y)) = \Im E \int_{-\infty}^y |\phi(s)|^2 ds \neq 0, \quad \forall y \in \mathbb{R}, \quad (19)$$

where the integral in (19) exists bounded for the semiclassical behavior. Thus, we state the result:

Lemma 5 *Let us consider the level $E = E_n^{\pm}(\hbar)$, for $\hbar < h_n$, so that $\Im E \neq 0$. The corresponding state $\psi(z) = \psi_n^{\pm}(\hbar, z)$ is different from 0 for all $z = iy$, $y \in \mathbb{R}$.*

5 Confinement of the nodes for large $\hbar > 0$

We prove a confinement of the nodes for the states in the case of a degenerate double well. Let us consider the level $\hat{E}_m(0)$, $m \in \mathbb{N}$, of $K(\alpha)$ at $\alpha = 0$, corresponding to the level $E_m(\hbar)$ of H_{\hbar} at the limit of $\hbar = +\infty$, because of the relation (6), $E_n(\hbar) = \hbar^{6/5} \hat{E}_n(-\hbar^{-4/5})$. It is relevant that the scaling used for this relation (6) is a regular one with a positive scale $\lambda = \hbar^{2/5}$ (even if infinite) respecting the angles on the complex plane. It is known that the level $\hat{E}_m(\alpha)$ is positive for $\alpha \geq 0$ [1]. We prove now a confinement of the zeros which allows us to distinguish the nodes from the other zeros.

We consider the operator $K(0)$ (3) translated by $x \rightarrow x + iy$,

$$K_y(0) = p^2 + i(x + iy)^3 = p^2 + i(x^2 - 3y^2)x + y^3 - 3yx^2 := p^2 + V_y(x) \quad .$$

We apply the Loeffel-Martin method [19] to a level $E = \hat{E}_m(0)$, with $E > 0$:

$$\begin{aligned} -\Im[\bar{\psi}(x+iy) \partial_x \psi(x+iy)] &= \int_x^{\infty} \Im V_y(s) |\psi(s+iy)|^2 ds = \int_x^{\infty} (s^2 - 3y^2) s |\psi(s+iy)|^2 ds = \\ &= - \int_{-\infty}^x (s^2 - 3y^2) s |\psi(s+iy)|^2 ds \neq 0, \end{aligned}$$

for $\pm x \geq \sqrt{3}|y|$, $y \in \mathbb{R}$. In this case we have a rigorous confinement, extendible to all $\alpha > 0$, of the region of the nodes,

$$C_{\sigma} = \{z = x + iy; y < 0, |x| < -\sqrt{3}y\} \subset C_{-} = \{z = x + iy; y < 0\}.$$

Since the same confinement extends to all $\alpha > 0$, we have that the m zeros of the state $\hat{\psi}_m(\alpha)$ on C_- are stable in the limit $\alpha \rightarrow +\infty$, i. e. are nodes by definition. Previous computations of the nodes [17] suggest that the present confinement may be sharp.

Thus, we state a result:

Lemma 6

All the nodes of the state $\psi_m(\hbar, z)$, $m \in \mathbb{N}$, for $\hbar > 0$ large enough, are in C_- .

6 The quantization rules

Suppose the existence of a continuation of each level $E^\pm := E_n^\pm(\hbar)$ from $\hbar < h_n$ to $\hbar \geq h_n$. For the moment, we keep the same names $E_n^\pm(\hbar)$ for the continuations of the levels even if such names are no more specific. We have two kinds of quantization rules for a fixed $\hbar > 0$ small, giving the eigenvalues $E_n^\pm(\hbar)$, respectively. Exist two regular circuits

$$\gamma^\pm,$$

such that, $P_x \gamma^+ = \gamma^-$,

$$\gamma^\pm = \partial D^\pm,$$

where D^\pm is a regular region large enough, with

$$D^\pm \subset C^\pm := \{x + iy, \pm x > 0, y \in \mathbb{R}\};$$

and,

$$\frac{1}{2i\pi} \oint_{\gamma^\pm} \frac{\psi'(\hbar, E^\pm, z)}{\psi(\hbar, E^\pm, z)} dz = n, \quad (20)$$

where $\pm \Im E^\pm \geq 0$. In particular, for small $\hbar > 0$, we have the semiclassical quantization condition,

$$\frac{1}{2i\pi} \oint_{\gamma^\pm} p_0(E^\pm, z) dz = \hbar(n + \frac{1}{2}) + O(\hbar^2). \quad (21)$$

This quantization conditions are still valid for all $\hbar > 0$, but, for large $\hbar \geq h_n$, are both satisfied by both the new states $\psi_m(\hbar)$, $m \in \mathbb{N}$, $[m/2] = n$ and the critical state ψ_n^c . We now prove that the state ψ_n^c has the set of $2n$ nodes

in $\mathbb{C}^+ \cup \mathbb{C}^-$. At the limit of $\hbar \rightarrow h_n^-$, the energy levels $E_n^\pm(\hbar)$ have the limit $E_n^c > 0$ and the set of the zeros of $\psi_n^\pm(\hbar)$ becomes P_x -symmetric. Let a node $N_j \in \mathbb{C}^+$ of ψ_n^+ to have a limit N_j^c as $\hbar \rightarrow h_n^-$. Because of the P_x symmetry of the set of all the zeros at the limit $\hbar \rightarrow h_n^-$, does exist a zero Z_k such that $Z_k \rightarrow P_x N_j^c$ as $\hbar \rightarrow h_n^-$. It is possible to disprove the possibility that $\Re N_j^c = 0$. Actually, in this case N_j^c would be a double zero of the state $\psi_n^c := \psi_n^\pm(h_n^-)$, but we know that the zeros are simple. Thus, the sets $S_n^\pm(\hbar)$ of the n nodes of the states $\psi_n^\pm(\hbar)$ have limits $S_n^\pm(h_n^-) \in \mathbb{C}^\pm$, respectively, as $\hbar \rightarrow h_n^-$. Both the states $\psi_m(\hbar)$, $m \in \mathbb{N}$, $[m/2] = n$ for $\hbar > h_n$, have $2n$ nodes in \mathbb{C}^{+-} . Actually, these nodes are stable: cannot become imaginary for the symmetry and the simplicity of the spectrum and cannot diverge along the imaginary axis. We have proved the:

Lemma 7

The critical state ψ_n^c , has a P_x -symmetric set of $2n$ nodes in $\mathbb{C}^{+-} := \mathbb{C}^+ \cup \mathbb{C}^-$,

$$S_n^c = S_n^\pm(h_n^-) \bigcup P_x S_n^\pm(h_n^-) = S_n^+(h_n^-) \bigcup S_n^-(h_n^-). \quad (22)$$

Both the states $\psi_m(\hbar)$, $m \in \mathbb{N}$, $[m/2] = n$ for $\hbar > h_n$, have $2n$ nodes in \mathbb{C}^{+-} .

In order to select a single state it is sufficient to use the inequality $E_{2n+1}(\hbar) > E_{2n}(\hbar)$. One more node of $\psi_{2n+1}(\hbar)$ lies on the imaginary axis. This is clear for large $\hbar > 0$, where all the nodes have a negative imaginary part. In this case it is possible to give a unique quantization rule,

$$\frac{1}{2i\pi} \oint_{\Gamma} \frac{\psi'(\hbar, E, z)}{\psi(\hbar, E, z)} dz = m, \quad (23)$$

where $m = 2n$ or $2n + 1$, and $\Gamma = \partial\Omega$ for $\Omega \subset \mathbb{C}_-$, $\mathbb{C}_- = \{z \in \mathbb{C}; \Im z < 0\}$. For $\hbar > 0$ large, it is convenient to use the scaling of the operator $K(\alpha)$ in order to have energy and nodes uniformly bounded.

Lemma 8

For each $n \in \mathbb{N}$, does exist $h_n > 0$ and a crossing:

$$E_n^\pm(h_n^-) = E_m(h_n^+) := E_n^c > 0,$$

$$\psi_n^\pm(h_n^-) = \psi_m(h_n^+) := \psi_n^c, \quad \forall m \in \mathbb{N}, \quad [m/2] = n. \quad (24)$$

Proof

The existence of this crossing is necessary because of the positivity of the

analytic functions $E_m(\hbar)$ for large $\hbar > 0$, and the non reality of the analytic functions $E_n^\pm(\hbar)$ for small $\hbar > 0$. The relation between the integer n and the integers m is due to the doubling of the nodes at h_n^- (Lemma 4) because of the P_x symmetry of the set of nodes. This nodes are in C^{+-} and are stable for $\hbar \geq h_n$. The differentiation of the number of nodes of the two states is necessary for large $\hbar > 0$. Actually, for $\hbar > h_n^o > h_n$ $\psi_{2n+1}(\hbar)$ has imaginary node $N_0 = iy$ with $y < \tilde{y}$, where the imaginary turning point is $I_0 = i\tilde{y}$.

Conjecture 1

We assume that the crossing between the pair E_n^\pm , $\forall n \in \mathbb{N}$, is unique.

This conjecture is justified by numerical, semiclassical and exact semiclassical results [10] [14], [12]. This conjecture allow us to simplify the notations. Now we prove that each level is bounded for bounded parameter $\hbar > 0$.

Lemma 9

Let $E(\hbar)$, be any level of the pair $E_n^\pm(\hbar)$, for $\hbar < h_n$, or any level of the pair $E_m(\hbar)$, $m \in \mathbb{N}$, $[m/2] = n$ for $\hbar > h_n$. Does not exists a $h^c \geq 0$, such that $E(\hbar)$ diverge as $\hbar \rightarrow h^c$.

Proof

We prove by absurd, and we consider the case $E = E_m(\hbar)$ for a fixed $m \in \mathbb{N}$, and \hbar near $h^c > 0$. The extension to the general case is simple. We consider the operator

$$\frac{H_\hbar - E(\hbar)}{|E|(\hbar)} \sim \hbar^2 p^2 + ix^3 - i\alpha x - \eta,$$

by a scaling $x \rightarrow \lambda x$, $\lambda = |E|^{1/3}$, where $\dot{\hbar} = \hbar|E|^{-5/3}$, $\alpha = |E|^{-2/3}$, $\eta = E/|E|$, $|\eta| = 1$. For small $\dot{\hbar} > 0$, by simply putting $\alpha = 0$, we have the semiclassical quantization condition,

$$\frac{1}{2i\pi} \oint_\Gamma \sqrt{\eta - iz^3} dz = \dot{\hbar}(m + \frac{1}{2}) + O((\dot{\hbar})^2). \quad (25)$$

which can be valid only if $\eta \rightarrow 0$ as $\dot{\hbar} \rightarrow 0$.

7 At the crossing the state is orthogonal to its P-transform

We have a crossing of $E_n^\pm(\hbar)$ at $\hbar = h_n$, when $\Im E_n^\pm(\hbar) = 0$. For $0 < \hbar < h_n$, the two clamped points of ψ_n^\pm are (I_\mp, I_0) respectively. At the crossing, we

have P_x symmetry of the turning points, so that $I_- = \bar{I}_+$, $I_0 = -\bar{I}_0$. Let $H := H_{\hbar}$, $H_{\hbar}^* = \bar{H} := H_{\bar{\hbar}}$, with two eigenvalues $E_j = \bar{E}_j$, and eigenvectors ψ_j $j = 1, 2$. We have

$$H\psi_1 = E_1\psi_1, \quad \bar{H}\bar{\psi}_2 = E_2\bar{\psi}_2,$$

so that

$$\langle \bar{\psi}_2, H\psi_1 \rangle = E_1 \langle \bar{\psi}_2, \psi_1 \rangle, \quad (26)$$

$$\langle H\psi_1, \bar{\psi}_2 \rangle = \langle \psi_1, \bar{H}\bar{\psi}_2 \rangle = E_2 \langle \psi_1, \bar{\psi}_2 \rangle,$$

and, by complex conjugation,

$$\langle \bar{\psi}_2, H\psi_1 \rangle = E_2 \langle \bar{\psi}_2, \psi_1 \rangle. \quad (27)$$

By subtraction of the two equations (26) (27), we get,

$$0 = (E_2 - E_1) \langle \psi_1, \bar{\psi}_2 \rangle.$$

Let now to vary the semiclassical parameter \hbar , so that:

$$0 = (E_2(\hbar) - E_1(\hbar)) \langle \psi_1(\hbar), \bar{\psi}_2(\hbar) \rangle,$$

for $\hbar > 0$. If $E_1(\hbar) \neq E_2(\hbar)$ for $\hbar > h_n$, and $E_1(h_n^+) = E_2(h_n^+) = E$, $\psi_1(h_n^+) = \psi_2(h_n^+) = \psi$, we have

$$0 = \langle \psi, \bar{\psi} \rangle = \langle \psi, P\psi \rangle = \int_{\mathbf{R}} \psi^2(x) dx. \quad (28)$$

Thus, we have proved the following:

Lemma 10

The PT -symmetric state at the crossing point,

$$\psi_n^c = \psi_{2n+1}(h_n^+) = \psi_{2n}(h_n^+) = PT\psi_n^c,$$

is completely P -asymmetric, i.e. is orthogonal to its P -transform:

$$\int_{\mathbf{R}} \psi_n^c(x)^2 dx = \langle \psi_n^c, P\psi_n^c \rangle = 0.$$

8 The Riemann surface near the real axis

Let us consider the sector on the \hbar complex plane (7):

$$C^0 = \{\hbar \in \mathbb{C}; \hbar \neq 0, \arg(\hbar) < \pi/4\},$$

and the Riemann sheet C_m^0 of the level $E_m(\hbar)$, $n = [m/2]$, defined in C^0 , with a square root singularity at h_n and a cut, $\gamma_{n,n} = (0, h_n]$. Thus, we prove the following:

Theorem 1

The levels $(E_{2n+1}(\hbar), E_{2n}(\hbar))$, $n \in \mathbb{N}$, are analytic functions defined on the Riemann sheets (C_{2n}^0, C_{2n+1}^0) , respectively, both with only the cut $\gamma_{n,n} = (0, h_n]$ on the real axis. The positive analytic functions $(E_{2n+1}(\hbar), E_{2n}(\hbar))$, with $E_{2n+1}(\hbar) > E_{2n}(\hbar)$ on $(h_n, +\infty)$ have the following values at the borders of the cut:

$$E_{2n}(\hbar \pm i0^+) = E_n^\pm(\hbar), \quad E_{2n+1}(\hbar \pm i0^+) = E_n^\mp(\hbar), \quad \forall 0 < \hbar < h_n. \quad (29)$$

Proof

Since both the functions $(E_{2n+1}(\hbar), E_{2n}(\hbar))$ have a square root singularity at h_n , and

$$E_{2n+1}(h_n + \epsilon) - E_{2n}(h_n + \epsilon) = O(\sqrt{\epsilon}) > 0,$$

for $\epsilon > 0$ small,

$$\pm \Im(E_{2n+1}(h_n + \exp(\pm i\pi)\epsilon) - E_{2n}(h_n + \exp(\pm i\pi)\epsilon)) < 0,$$

and $\mp \Im E_n^\pm(h) > 0$, for $h < h_n$, necessarily we have,

$$E_{2n+1}(h_n + \exp(\pm i\pi)\epsilon) := E_n^\mp(h_n - \epsilon),$$

$$E_{2n}(h_n + \exp(\pm i\pi)\epsilon) := E_n^\pm(h_n - \epsilon).$$

Remark

The Riemann sheet C_0^0 of the fundamental level has only one cut $\gamma_{0,0} = [0, h_0]$ on \mathbb{R} [12], and the discontinuity on the cut is defined by the rule,

$$E_0(\hbar \pm i0^+) = E_0^\pm(\hbar), \quad \forall \hbar, \quad 0 < \hbar < h_0. \quad (30)$$

We recall, for instance, that $E_0^+(\hbar) := E_0(\hbar + i0)$, is defined as the limit from above for small $\hbar > 0$. This definition extends directly to all $\hbar > 0$ in absence of complex singularities. Formula (30) means that it is possible the absence of other singularities involving the function $E_0(\hbar)$. Thus, using the principle of maximal analyticity, we assume that in C_0^0 there is only the cut $\gamma_{0,0}$.

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